Chapter 7 solution

7.12 $\sigma = 4$.

$P(|\bar{X} - \mu| < 1) \geq 0.90$

$\Rightarrow P(\mu - 1 < \bar{X} < \mu + 1) \geq 0.90$

$\Rightarrow P\left(-\frac{1}{4/\sqrt{n}} < \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} < \frac{1}{4/\sqrt{n}}\right) \geq 0.90$

$\Rightarrow P\left(-\frac{\sqrt{n}}{4} < Z < \frac{\sqrt{n}}{4}\right) \geq 0.90$

$\Rightarrow 1 - 2 \times P\left(Z \leq -\frac{\sqrt{n}}{4}\right) \geq 0.90$

$\Rightarrow P\left(Z \leq -\frac{\sqrt{n}}{4}\right) \leq 0.05$

$\Rightarrow -\frac{\sqrt{n}}{4} \leq -1.645$

$\Rightarrow n \geq 43.3 \quad \Rightarrow n = 44$

7.38 (a) $Y_6 \sim N(0,1)$, $W \sim \chi^2_5$, and $Y_6 \perp W$

$\Rightarrow \frac{\sqrt{5}\, Y_6}{\sqrt{W}} = \frac{Y_6}{\sqrt{W/5}} \sim t_5$

(b) $Y_6 \sim N(0,1)$, $U \sim \chi^2_4$, $Y_6 \perp U$

$\Rightarrow \frac{2 Y_6}{\sqrt{U}} = \frac{Y_6}{\sqrt{U/4}} \sim t_4$

(c) $\bar{Y} \sim N(0, \frac{1}{5})$, $Y_6 \sim N(0,1)$, $\bar{Y} \perp Y_6$

$U \sim \chi^2_4$, $U \perp Y_6$, $U \perp \bar{Y}$.

$$\Rightarrow \frac{2(5\bar{Y}^2 + Y_6^2)}{U} = \frac{\left[(\sqrt{5}\bar{Y})^2 + Y_6^2\right]/2}{U/4}$$

Note that $\sqrt{5}\bar{Y} \sim N(0,1)$ and $\sqrt{5}\bar{Y} \perp Y_6$

Hence $(\sqrt{5}\bar{Y})^2 + Y_6^2 \sim \chi^2_2$.

Therefore $\frac{2(5\bar{Y}^2 + Y_6^2)}{U} \sim F_{2,4}$.

7.37: (a) $W \sim \chi^2_5$

(b) $U \sim \chi^2_4$

Proof: $U = \sum_{i=1}^{5} (Y_i - \bar{Y})^2$

$$= \frac{(5-1) \times \frac{1}{5-1} \sum_{i=1}^{5} (Y_i - \bar{Y})^2}{1} \quad \left(\text{Hint: } \frac{(n-1)S^2}{\sigma^2}\right)$$

$$\sim \chi^2_4$$

(c) χ^2_5

7.42

(a) $P(\bar{Y} > 14.5)$

$= P\left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} > \frac{14.5-14}{2/\sqrt{100}}\right)$ $\left(\begin{array}{l} n=100>30, \\ \text{apply CLT} \end{array}\right)$

$\approx P(z > 2.5) = 0.006$

7.48 (a) $P(\mu-1 < \bar{X} < \mu+1)$

$= P\left(-\frac{1}{12/\sqrt{35}} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < \frac{1}{12/\sqrt{35}}\right)$

$\approx P(-0.49 < z < 0.49)$

$= 0.376$

(b) No, Population mean not necessarily equal to the sample mean.

7.52 (a) $P(199 < \bar{Y} < 202)$

$= P\left(\frac{199-200}{10/\sqrt{25}} < \frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} < \frac{202-200}{10/\sqrt{25}}\right)$

$\approx P(-0.5 < z < 1)$

$= 0.533$

(b) $P\left(\sum Y_i \le 5100\right)$

$$= P\left(\frac{\sum Y_i}{n} \le \frac{5100}{25}\right)$$

$$= P\left(\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \le \frac{204-200}{10/\sqrt{25}}\right)$$

$$\approx P(z \le 2) = 0.977$$

7.76(a) $Var(Y/n) = \frac{1}{n^2} Var(Y) = \frac{1}{n} p(1-p)$

$$\le \frac{1}{n} \cdot \frac{(p+(1-p))^2}{2} = \frac{1}{2n}$$

where "=" hold when $p = 1-p \Rightarrow p = 0.5$

(b) $P\left(p-0.1 < \frac{Y}{n} < p+0.1\right) \ge 0.95$

$$\Rightarrow P\left(-\frac{0.1}{\sqrt{p(1-p)/n}} \le \frac{\bar{X}-p}{\sqrt{p(1-p)/n}} < \frac{p+0.1-p}{\sqrt{p(1-p)/n}}\right) \ge 0.95$$

$$\Rightarrow P\left(-\frac{0.1}{\sqrt{p(1-p)/n}} < z < \frac{0.1}{\sqrt{p(1-p)/n}}\right) \ge 0.95$$

$$\Rightarrow 1 - 2P\left(z \le -\frac{0.1}{\sqrt{\frac{p(1-p)}{n}}}\right) \ge 0.95$$

$$\Rightarrow P\left(z \le -\frac{0.1}{\sqrt{\frac{p(1-p)}{n}}}\right) \le 0.025$$

$$\Rightarrow -\frac{0.1}{\sqrt{\frac{p(1-p)}{n}}} \leq -1.96$$

$$\Rightarrow n \geqslant (19.6)^2 p(1-p)$$